ISSN: 2148-9955

International Journal of Research in Education and Science (IJRES)

www.ijres.net

Middle School Students' Interpretation of Definitions of the Parallelogram Family: Which Definition for Which Parallelogram?

Nilüfer Yavuzsoy-Köse[1], Tuğba Y. Yilmaz[2], Demet Yeşil[2], Duygu Yildirim[2]
[1]Anadolu University
[2]Ministry of National Education

To cite this article:

Yavuzsoy Kose, N., Yilmaz, T.Y., Yesil, D., & Yildirim, D. (2019). Middle school students' interpretation of definitions of the parallelogram family: Which definition for which parallelogram? *International Journal of Research in Education and Science (IJRES), 5*(1), 157-175.

IJRES International Journal of Research in Education and Science
Volume 5, Issue 1, Winter 2019 ISSN: 2148-9955

Middle School Students' Interpretation of Definitions of the Parallelogram Family: Which Definition for Which Parallelogram?

Nilüfer Yavuzsoy-Köse, Tuğba Y. Yılmaz, Demet Yeşil, Duygu Yıldırım

Article Info

Article History

Received:
02 July 2018

Accepted:
08 November 2018

Keywords

Mathematics education
Learning geometry
Quadrilaterals
Parallelograms
Exclusive-inclusive definitions

Abstract

The aim of this study was to investigate how students interpret the verbal definitions given for quadrilaterals in parallelogram family and reasoning in this interpretation process. It was also aimed to reveal the easiest definitions for students to understand, and to determine which mathematical terms they do not understand in given definitions. The study sample consisted of 16 volunteer eighth grade students. Data were collected with the clinical interview method and analyzed with thematic analysis method. The results revealed that the exclusive definition and inclusive definition (based on sides) of a parallelogram, the exclusive definitions of a rectangle and a rhombus, and the definition of a square based on diagonals were identified more accurately by the students. Analysis of the students' reasoning process showed that the vast majority of the students made their decisions by comparing a given definition with other quadrilaterals and that few students explained their reasons to choose a definition directly. It was also determined that prototype quadrilateral images were effective in thinking process of the students who inaccurately interpreted definitions in this process. Finally, the students mostly had difficulty in understanding expressions such as rotational symmetry, bisecting, bisecting perpendicularly, having at least one angle of 90°, having one right angle, perpendicular bisector and symmetrical across the perpendicular bisector.

Introduction

Geometry, which is an important learning area of mathematics, mediates between mathematics and the real world. Geometry is a science that makes it easy for us to understand the environment where we live in terms of location, direction and shape; involves knowledge and relationships of many shapes and objects; makes it possible for individuals to perform reasoning, problem-solving, critical thinking and establishing cause-effect relationships; and equip individuals with higher-level thinking skills (Türnüklü & Berkün, 2013). Its axiomatic structure-in other words, the structure that requires using and interpreting postulates, axioms, definitions and theorems in thinking process-has placed geometry in a unique position in mathematics. Deductive nature of geometry helps individuals establish a solid logical system by strengthening the connections between concepts. One of the building blocks of this system is certainly definitions.

Definitions in mathematics include fundamental ideas that help us make sense of a word, a sentence or a symbol (Usiskin, Griffin, Witonsky, & Willmore, 2008). Definitions may be descriptive so as to include previously known characteristics of a concept or they may be constructive so as to provide a new concept definition constructed from a given definition. These two functions of definitions allow systematization of existing knowledge as well as the production of new knowledge (de Villiers, 1998). However, definitions, which are effective tools for systematizing existing knowledge and building new knowledge, also involve challenges for both students and teachers. As they are developed based on concise, intuitive, necessary and sufficient conditions so as to reflect the characteristics of a concept, definitions include many technical terms that are inherent in nature (de Villiers, Govender & Patterson, 2009). Formal or informal use of these technical terms in the classroom setting also influences the process of understanding/making sense of definitions, which indicates how important mathematical language is in teaching environments. Mathematical terms consist essentially of words, and Rubenstein and Thompson (2002) found that students have some common difficulties in learning mathematical words. Rubenstein and Thompson (2002) also stated that although some English words are present in everyday life and mathematics, they do have different meanings, some English words and mathematical words have similar meanings but mathematical meanings can be more specific, some mathematical words are used only in mathematics and some words have more than one meaning. As one can imagine, this situation is not limited to only English words. Some mathematical words in Turkish are confused with their everyday uses, too. The best example of this situation is the word "trapezoid (*yamuk* in Turkish)". Ulusoy (2015) found that students were confused about the use the Turkish equivalent of a trapezoid in daily life and a trapezoid that is actually a quadrilateral, and that they identified all polygons with more than four sides or shapes that do not consist of line segments as trapezoids. This indicates that students are familiar with the daily use of the term but do not know the mathematical meaning or even realize the distinction between them (Monroe & Panchyshyn, 1995). Another example is the fact that students cannot distinguish between words such as "parallelism and perpendicularity", which are often used together but have different meanings (Casa & Gavin, 2009). There are also study results showing that students tend to focus on the concept of "parallel sides" rather than the concept of "parallelograms", and they treat any shape (e.g. hexagon or two parallel lines) with parallel sides as parallelograms (Ulusoy & Çakıroğlu, 2017). All these results show that cautious and careful selection of mathematical terms to be used in the classroom or in textbooks for definitions is an imperative requirement for mathematical comprehension. The fact that there are different definitions of quadrilaterals, especially at middle school level, makes it more important to investigate how students make sense of and interpret these definitions. Therefore, the aim of this study was to examine how students interpret the definitions verbally given for parallelograms, rectangles, rhombuses and squares, and which ways of reasoning they use in this interpretation process. The study also aimed to determine the easiest definitions for students to understand and to find out what mathematical terms they had difficulty in understanding in the definitions given.

What is a Definition and What are Different Forms of Definition?

Quite many studies investigated the effect of visualization on the construction of geometrical shapes (Vinner & Herskowitz, 1980; Fujita &Jones, 2007). Researchers focused on cognitive construction of mathematical concepts, and proposed a model of two components: The *concept definition*-the verbal description of the mathematical concept, which characterizes the concept mathematically, and the *concept image*-the cognitive structure that includes all the examples and the processes related to the concept in the learner's mind (Vinner & Hershkowitz, 1980; Tall&Vinner, 1981; Haj-Yahya & Hershkowitz, 2013). Researchers also found that for each geometric concept, learners had at least one prototype example. Prototypical examples in the concept image contains examples of some of the features included in the longest feature list of the concept and causes the learner to have a personal perception of a limited concept about a figure.

Many other studies focus on definition in geometry (Zaslavsky & Shir, 2005; de Villiers et al., 2009). Definitions are vital for concept development in geometry. The knowledge acquired in line with assumptions in definitions that play a critical role in geometry is organized and the image of a shape is visualized mentally by building new knowledge on this accumulation of knowledge (de Villiers, 1994). Definitions are tools that are often used to better understand a particular concept, and because they act as the foundation stone of mathematical thinking, they also act as a base for the formation of a mathematical concept, distinguishing it from other concepts, and expressing mathematical thoughts. Correctly defining a concept takes more than just listing the properties of the concept. As matter of fact, it should be possible to specify the concept by making logical inferences from the properties of the concept and to choose the properties that best describe the concept with necessary and sufficient conditions (Çakıroğlu, 2013). According to Jamison (2000), a definition is a clear description of the basic properties of an object or concept with few and specific expressions. In a mathematical definition, it is necessary to use at least the required properties instead of listing all known properties of the concept. This situation involves a definition being as economical as possible. For example, instead of defining a rectangle as a quadrilateral with opposite sides of equal length and four angles of 90°, a rectangle can simply be defined as a quadrilateral with three right angles. In addition, even under different representations, a mathematical definition must be invariant, hierarchical and based on previously defined concepts or basic concepts (Zaslavsky & Shir, 2005).

Students often come across with definitions in mathematics textbooks. In these books, there is usually only one correct definition for each concept defined. On the other hand, there are a variety of correct definitions for a particular concept, and both teachers and students should have the freedom to choose from these definitions. For example, a rectangle can be defined as a parallelogram with a right angle, or it can be defined as a quadrilateral with three right angles or as a quadrilateral with sides symmetrical about the perpendicular bisector (de Villiers et al., 2009). In the light of this, each definition for a parallelogram, a rectangle, a rhombus and a square addressed in this study is presented along with alternative definitions based on side, angle, diagonal and symmetry properties. The idea here is inferring other properties of any given quadrilateral based on the symmetry-based definitions by selecting symmetry property as well as side, angle and diagonal properties. For example, when we compare two definitions of rhombuses "a quadrilateral that is symmetric across both diagonals" and "a quadrilateral with all sides congruent", we can suggest that the definition based on symmetry is more deductive and economical. This definition makes it possible to deduce other properties such as a rhombus having diagonals that are perpendicular bisectors of each other and all sides being congruent (de Villiers et al., 2009).

There is a common relationship between definition and classification. The presence of inclusive and exclusive definitions of quadrilaterals (Usiskin et al., 2008) leads to the presence of different classifications. While both inclusive and external definitions are correct, inclusive definitions produce a hierarchical classification of quadrilaterals, but exclusive definitions classify quadrilaterals with segments independent of each other. Verbal expressions in inclusive definitions of quadrilaterals can indicate one or more categories of shapes beyond the specified quadrilateral. Therefore, the resulting hierarchical relationships become more economical and functional (Fujita & Jones, 2007). For example, an inclusive definition of a rhombus can be made as "a quadrilateral with four equal sides". This definition implies that "a square is a special type of a rhombus". This inclusivity implies that a property that is valid for a rhombus also applies to a square. The definition of a rhombus as "a quadrilateral with four equal sides and no right angles" can be taken as an exclusive definition. Therefore, a square cannot be recognized as a special type of a rhombus according to this definition. Establishing relationships among quadrilaterals through these classifications is crucial for students' cognitive development, helps them improve their geometric thinking skills and can be considered as an introduction to the deductive structure of geometry (Okazaki & Fujita, 2007; Fujita, 2008). In the light of these considerations, this study addressed both inclusive and exclusive definitions of quadrilaterals and examined how students interpret the quadrilateral classifications in inclusive definitions.

Inclusive and exclusive definitions of quadrilaterals lead to various relationships among concepts. Some studies about quadrilaterals (Tall & Vinner, 1981, de Villiers, 1994, Heinze & Ossietzky, 2002, Fujita & Jones, 2007, Kondratieva & Radu, 2009, Aktaş & Aktaş, 2012, Fujita, 2012, Ulusoy & Çakıroğlu, 2017) showed that middle school students had difficulty in defining quadrilaterals and making hierarchical classification. Moreover, these studies also found that use of inclusive or exclusive definitions in defining quadrilaterals influenced establishing relationships between quadrilaterals. Some studies, on the other hand, found that students' difficulty in comprehending the inclusive relationships varied depending on their grade levels and these difficulties were associated with possession of strong prototypes of shapes and properties of shapes (Okazaki & Fujita, 2007; Güven & Okumuş, 2011; Jones & Tzekaki, 2016). Ulusoy and Çakıroğlu (2017), for example, asked middle school students to make verbal and written definitions of parallelograms first and then to draw different parallelogram examples and to identify parallelograms among shapes that were parallelograms and that were not, and they found that prototypes with non-hierarchical or partial hierarchical properties had effects on students' sample spaces regarding parallelograms. Heinze and Ossietzky (2002) argued that this difficulty is related to understanding the mathematical language and necessary and sufficient conditions in definitions, as well as the idea of classification of quadrilaterals. Haj-Yahya and Hershkowitz (2013), on the other hand, found that 10^{th} grade students had a more accurate understanding and explanation of the inclusive quadrilateral relationships based on verbal definitions that were given without any visuals. In fact, the authors also stated that the visual properties of quadrilaterals given affected students' judgment and classifications, and that these effects were hidden in verbal definitions. The starting point of this study lies on the question of whether this significant finding applies to 8^{th} grade students as well. Since this study focuses on how students can interpret inclusive and exclusive definitions, the emphasis was on students' recognition of given quadrilaterals and the members of the parallelogram family which students often encounter as quadrilaterals beginning from elementary school (i.e. quadrangle, rectangle, rhombus and square). Furthermore, there is currently limited research exploring how students interpret inclusive (e.g., side, angle, diagonal and symmetry properties) and exclusive definitions of quadrilaterals given verbally and without visual support or determining which definitions of quadrilaterals they find easier to understand based on the given properties. On the other hand, it is important for mathematics teachers and mathematics educators to figure out how students interpret a quadrilateral definition, to discover the reasoning behind this interpretation and to choose appropriate definitions in their teaching. In this respect, investigating how students interpret the verbal definitions given for a parallelogram, a rhombus, a rectangle and a square, how they perform reasoning in this process and which expressions students have difficulty in understanding in this process could contribute to the relevant literature.

Method

A basic qualitative research approach was adopted in this study in order to examine how students interpret the concept of quadrilaterals in general, and a parallelogram in particular, and to determine which given definitions of parallelograms (including a rhombus, a rectangle and a square) they find easier to understand. The basic qualitative research aims to explore individuals' perspectives and worldviews, or to explore, understand and make sense of a process (Merriam, 2009). Therefore, in this study, various definitions of a parallelogram were presented to the students in this context and how they interpreted these definitions were determined.

Participants

The participants of the study were 16 volunteer eighth grade students in two different public middle schools. This study was conducted in a city in Central Anatolia Region, Turkey, the socio-economic status of the students at the selected schools was similar and their average achievement levels were close to each other (compared to the national examination results in the previous year during). The participants were determined using the criteria sampling method, a purposeful sampling method. According to the inclusion criteria, the participants would be eighth grade students and know definitions of parallelograms (including a rhombus, a rectangle and a square). In Turkey, students are normally introduced to the basic properties of a trapezoid and the parallelogram family beginning from the 5th grade. Regarding the inclusive relationships, mathematics curriculum covers a square and a rectangle as special parallelograms in 6th grade and a square as a special rectangle and a special rhombus, a rectangle and a rhombus as special parallelograms, and a rectangle, rhombus and a parallelogram as special trapezoids in 7th grade (MoNE, 2013). Because the aim of this study was to question students' knowledge about definitions and to understand how they interpret the definitions given, the criterion that students should already have knowledge of the definitions and properties of quadrilaterals made it necessary to choose the participants among eighth grade students. For this reason, about 200 eighth grade students were asked questions about definitions of parallelograms, and 16 students with the most accurate verbal expressions among those who made the most accurate definitions were selected as the participants for this study. The eight participants from the first public school were coded as S1 -...- S8 and the other eight participants from the second public school were coded as S9-… -S16.

Data Collection

Data were collected with the clinical interview method, which is a type of interview technique used frequently in research on mathematics education (Clement, 2000). Clinical interviews provide elaborate observation of students' thinking structures and cognitive processes (Ginsburg, 1981; Clement, 2000). Taking into account the aim of the study, after the inclusive and exclusive definitions of a parallelogram, a rectangle, a rhombus and a square were determined, the students were asked to interpret these definitions during the clinical interviews. The inclusive and exclusive definitions were determined based on the related literature (Usiskin et al., 2008; Öztoprakçı & Çakıroğlu, 2013) and at least one inclusive definition that included side, angle, diagonal and symmetry properties for each quadrilateral was included. Parallelogram, rectangle and rhombus were each given five definitions. However, since there is no inclusive definition of square, four definitions were chosen for square based on side, angle, diagonal and symmetry properties and no additional exclusive definition was given for a square. Table 1 shows a total of 19 definitions determined for a parallelogram, a rectangle, a rhombus and a square. As can be seen in Figure 1, the students were given only the written definitions in Table 1 and they were asked to identification the quadrilateral(s) each definition described, and finally they were asked to explain and justify their answer(s).

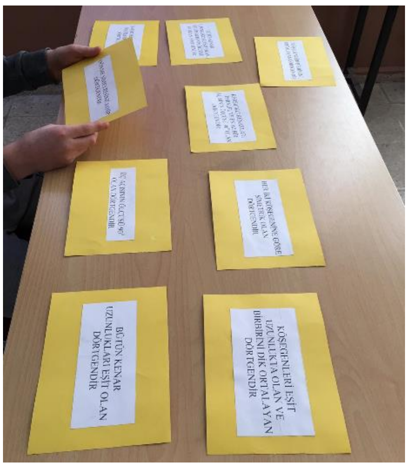


Figure 1. A sample clinical interview session

The clinical interviews were conducted in two sessions in a quiet environment where students could feel comfortable, and the interviews were recorded. The students were asked questions about 9 to 10 verbal definitions in each session The students were given enough time to explain their ideas, and each interview session lasted 35-45 minutes.

Table 1. The quadrilateral definitions given to the students

Quadrilateral	Definition	Definition property
Parallelogram	A quadrilateral with two pairs of parallel sides (Par^1).	Side
	A quadrilateral with opposite angles equal. (Par^2).	Angle
	A quadrilateral whose diagonals bisect each other (Par^3).	Diagonal
	A quadrilateral that has rotational symmetry (Par^4).	Symmetry
	A quadrilateral whose diagonals bisect each other, are not equal in length and are not perpendicular bisectors of each other (Par^5).	Exclusive
Rectangle	A quadrilateral with opposite sides parallel and at least one angle of 90° (Rec^1).	Side
	A quadrilateral with three right angles (Rec^2).	Angle
	A quadrilateral with diagonals that are congruent and bisect each other (Rec^3).	Diagonal
	A quadrilateral with sides symmetrical to the perpendicular bisector (Rec^4).	Symmetry
	A quadrilateral with one angle of 90° and opposite sides parallel but adjacent sides not equal in length (Rec^5).	Exclusive
Rhombus	A quadrilateral with four equal sides (Rh^1).	Side
	A quadrilateral with opposite angles equal and all sides equal length (Rh^2).	Angle
	A quadrilateral in which the diagonals are perpendicular bisectors of each other (Rh^3).	Diagonal
	A quadrilateral that is symmetric across both diagonals (Rh^4).	Symmetry
	A quadrilateral with four equal sides and no right angles (Rh^5).	Exclusive
Square	A quadrilateral with equal sides and one right angle (Squ^1).	Side
	A regular quadrilateral (Squ^2).	Angle
	A quadrilateral with diagonals that are congruent and perpendicular bisectors of each other. (Squ^3).	Diagonal
	A quadrilateral that is symmetrical to the diagonals and perpendicular bisectors (Squ^4).	Symmetry

Data Analysis

Data were analyzed using the thematic analysis method. In this method, the themes used are concepts derived from research data. The themes could be determined based on the expressions participants use during the interview or by the researcher identifying the information embedded in the data based on his or her expertise in the field. In this respect, the data were coded independently by four mathematics education experts, who are the authors of this paper, and the relevant themes and sub-themes were organized as an interrelated and meaningful whole. The data analysis process was carried out in two steps. In the first step, the clinical interview data were transcribed, and the data were read out repeatedly so that the researchers could make sense of the thoughts expressed by the participants. The second step involved coding and theme development processes. The emerging themes, sub-themes and categories were presented in detail in the coding key given in Table 2.

In the analysis process, two main themes were determined based on the students' responses about definition-quadrilateral matching: "Identification process" and "Reasoning process". The theme of identification process was divided into three sub-themes: "Accurate identification," "Inaccurate identification" and "Lack of comprehension of a definition". The correct answers given by the students included giving the correct answer directly, giving the correct answer with the researcher's guidance and taking into account the quadrilateral covered by the definition in addition to giving the correct answer. Therefore, the sub-theme of accurate identification was divided into three categories: "Accurate identification," "Accurate identification with guidance" and "Considering the quadrilateral covered".

For the sub-theme of inaccurate identification, the category of students with "Inaccurate identification" was given first. After that, the reasons for inaccurate identification were presented under two categories: "Failure to recognize based on the inclusive definition" and "Considering only the quadrilateral covered". Failure to distinguish based on the inclusive definition refers to students' inability to identify the most general quadrilateral based on a given definition and to perform match definitions and quadrilaterals accurately taking into account the quadrilateral covered in the definition. Considering only the quadrilateral covered refers to students' thinking only the most specific type in the definition given or, in other words, considering only the quadrilateral(s) covered by the inclusive definition.

Table 2. Coding key and descriptions

Themes		***Sub-themes***		***Explanations and Students Samples***
Identification Process	**Accurate**	***Accurate identification***		Some students accurately determined which quadrilateral a given verbal definition described. For example, a student said, "A quadrilateral with two pairs of opposite sides parallel." is parallelogram.
		Accurate identification with guidance		Some students accurately determined which quadrilateral a given verbal definition described based on explanations made by the teacher. For example, the teacher helped a student who were confused about the definition "A quadrilateral whose diagonals bisect each other" by saying, "Think about it, these are definitions for quadrilaterals, not just their properties. So when you read a definition, that definition should be sufficient to identify a quadrilateral. It will not require anything else" or by giving hints when a student had difficulty by asking, "If it is a quadrilateral with three right angles, can you calculate the 4th angle?"
		Considering the quadrilateral covered		Some students determined the quadrilaterals described by the given verbal inclusive definitions. For example, realizing that the definition "A quadrilateral with two pairs of opposite sides parallel" is an inclusive definition of parallelograms, a student stated that it should also valid to "a rectangle, a square and a rhombus".
	Inaccurate	***Inaccurate identification***		Some students were unable to determine which quadrilateral the given verbal definition described. For example, for the definition "A quadrilateral that is symmetrical across both diagonals", a student suggested a parallelogram by saying, "a parallelogram is also symmetrical because it is bisected equally by two diagonals."
		Reasons for inaccurate identification	*Failure to recognize based on the inclusive definition*	Some students were unable to recognize the most general quadrilateral based on the given verbal definition and to recognize which quadrilateral the definition described adequately considering the quadrilateral included as well. For example, for the definition "A quadrilateral whose diagonals bisect each other", a student was not able to recognize the quadrilateral it defined by saying, "this applies to all of them because they bisect each other in all of them."
			Considering only the quadrilateral covered	Some students thought of the most specific case only, in other words the quadrilateral described by the inclusive definition, for the given verbal definition. For example, for the definition "A quadrilateral with sides symmetrical to the perpendicular bisector", a student suggested only a square instead of a rectangle by saying, "this definition applies to a square, because it bisects symmetrically, so I think it is a square."
	Lack of comprehension of a definition			Some students were unable to understand the given verbal definition. For example, for some definitions, some students said, "I do not understand what 'bisecting each other' means. I do not get the question" or "I do not know what 'a regular quadrilateral' means".
Reasoning Process		*Reasoning based on comparison*		Some students analyzed the given verbal definition in comparison with other quadrilaterals. Some other students reached a conclusion by testing the given definition for all the quadrilaterals they considered. For example, for the definition "A quadrilateral with sides symmetrical to the perpendicular bisector", a student said, "They are not symmetrical to the perpendicular bisectors in a rhombus. It does not valid to a parallelogram, either. Then this must be a rectangle."
		Reasoning based on justification		When interpreting a given verbal definition, some students came up with an idea without making any comparison but directly based on the definition and properties. For example, a student concluded that the definition must be describing a parallelogram simply because of the congruency of angles in a quadrilateral with opposite sides parallel.
		Inaccurate reasoning based on prototype quadrilateral image		For some of the students, the quadrilateral image caused them to misinterpret a definition. For example, for the definition "A quadrilateral that has rotational symmetry", a student said, "When we rotate a square, it becomes a rhombus, and when we rotate it again, it becomes itself again."

The students' thinking processes were analyzed under the theme of "Reasoning process" that consisted of three types of reasoning: "Reasoning based on comparison," "Reasoning based on justification" and "Inaccurate reasoning based on prototype quadrilateral image". Reasoning based on comparison refers to students' analysis of a definition in comparison with other quadrilaterals in the interpretation process. Reasoning based on justification refers to students' direct identification without any comparison and presentation of a thought based on relevant properties while interpreting a definition. Inaccurate reasoning based on prototype quadrilateral image implies that the quadrilateral image in students' minds causes a misinterpretation of a definition.

The four researchers developed codes and themes independently. The analysis results for these codes and themes indicated a confidence level of 90% (Miles and Huberman, 1994). The emerging themes, sub-themes and categories were presented in tables and diagrams in the results section, and the students were coded as $S_{1,}$ $S_{2,\ldots.}S_{16}$ to indicate their reasoning processes.

Results

The students' interpretations of the definitions given and their reasoning processes in their interpretations were presented separately for each of the quadrilaterals.

Identification and Reasoning Process of the Definitions of a Parallelogram

Table 3 shows the students' identification of the definitions of a parallelogram among the definitions given.

Table 3. Students' identification of parallelogram definitions

		Accurate identification			**Inaccurate identification**			**Lack of comprehension of a definition**
					Inaccurate Identification	*Reasons for Inaccurate Identification*		
		Accurate identification	*Accurate identification with guidance*	*Considering the quadrilateral covered*		*Failure to recognize based on the inclusive definition*	*Considering only the quadrilateral covered*	
Par[1]		15 (93,75)	0	14 (87,5)	1 (6,25)	1 (6,25)	0	0
Par[2]		8 (50)	6 (37,5)	12 (75)	2 (12,5)	0	2 (12,5)	0
Par[3]	f(%)	2 (12,5)	0	2 (12,5)	12 (75)	6 (37,5)	6 (37,5)	2 (12,5)
Par[4]		0	0	0	13 (81,25)	2 (12,5)	11 (68,75)	3 (18,75)
Par[5]		13 (81,25)	2 (12,5)		1 (6,25)	0	1 (6,25)	0

As can be seen in Table 3, nearly all of the students accurately identified the definition "A quadrilateral with two pairs of parallel sides (Par[1])" as a parallelogram. Only one of the students (S_2) had difficulty understanding the phrase "two pairs of parallel sides" in the definition. It was also observed that the vast majority of these students recognized that this definition was inclusive and included rectangles, rhombuses and squares, too. The following dialogue from the interview with one of the students could be given as an example of this situation:

S_8	: A quadrilateral with two pairs of parallel sides. A parallelogram.
Researcher (R)	: Why do you think so?
S_8	: It is a parallelogram because they (sides) are parallel.
R	: What about the others? Do they not have two pairs of parallel sides?
S_8	: A square, a rectangle and a rhombus have parallel sides, too. But this definition may be sufficient for a parallelogram.

The vast majority of the students identified the inclusive definition "A quadrilateral with opposite angles equal (Par[2])" based on angle properties and the exclusive definition "A quadrilateral whose diagonals bisect each other, are not equal in length and are not perpendicular bisectors of each other (Par[5])". In particular, 75% of the students who identified Par[2] as a parallelogram said that this definition was valid for the quadrilaterals it covered. We observed that the two students who identified the definition Par^2 incorrectly stated that the description only indicated a square, a rectangle and a rhombus, while the student who interpreted Par^5 incorrectly stated that the definition described a rectangle.

Another notable finding in Table 3 is that the students failed to identify the inclusive definition "A quadrilateral whose diagonals bisect each other (Par^3)" and the inclusive definition "A quadrilateral that has rotational symmetry (Par^4)" based on symmetry property as a parallelogram. There were also students who did not understand these two definitions. While some students had difficulty in understanding the expression "bisect" in the definition Par^3, some other students did not understand the concept of "rotational symmetry" in the definition Par^4. Also, some students failed to recognize a parallelogram based on the inclusive definition Par^3 and others stated that the definition might be valid for quadrilaterals covered by a parallelogram (a square by two students, a rhombus by one student, a square and a rhombus by two students, and a square and a rectangle by one student) but not for a parallelogram. The following dialogue from the interview with one of the students who failed to recognize the shape can be presented as an example of this situation:

S_4 : A quadrilateral whose diagonals bisect each other. They bisect each in a rectangle, in a square and in a rhombus (thinking about a parallelogram). They bisect each other in it, too (for parallelogram). It is a definition for all four of them.

R : Think about it, these are definitions for quadrilaterals, not just their properties. So when you read a definition, that definition should be sufficient to identify a quadrilateral. It will not require anything else.

S_4 : I think they all fit the definition.

As can be seen in the dialogue above, the student was able to tell which quadrilaterals the given verbal definition applied to but could not decide for which quadrilateral it was an adequate definition. The majority of the students who incorrectly identified the definition Par^4 focused on the quadrilaterals covered by a parallelogram and stated that the only quadrilaterals with rotational symmetry were "a square" and "a rhombus". In the students' processes of reasoning in interpreting definitions given for a parallelogram, it was seen that they mainly used three ways and these processes are presented in Figure 2. The students shown in red are those who misinterpreted the given definition.

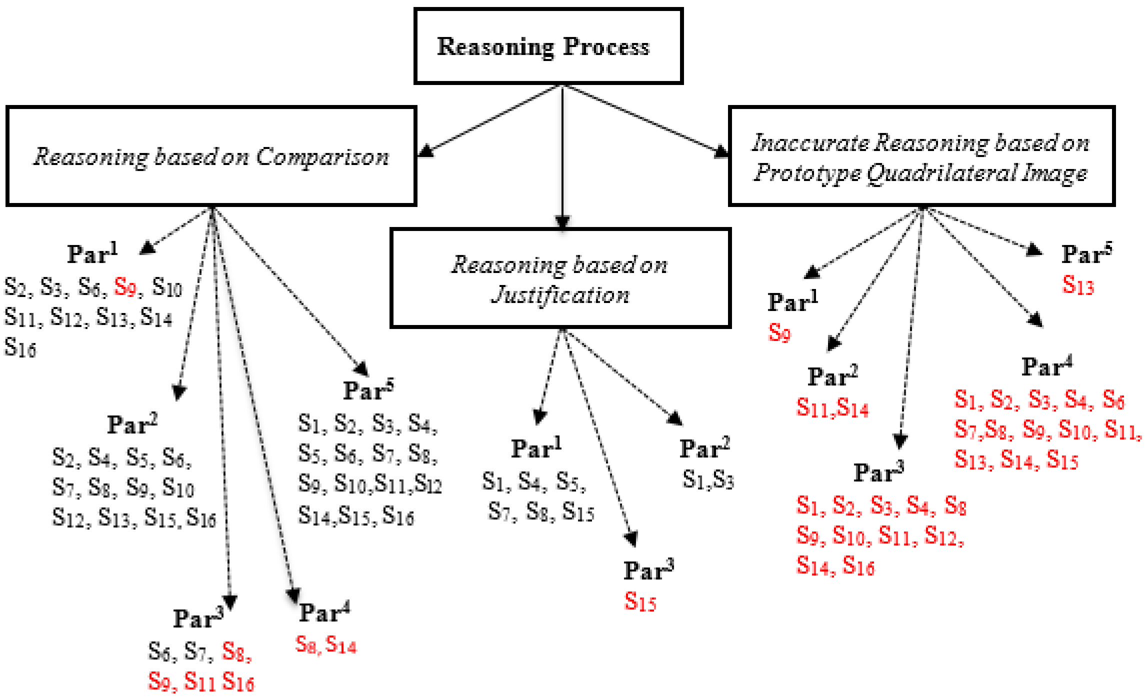


Figure 2. The students' processes of reasoning in definitions given for a parallelogram

As can be seen in Figure 2, the students primarily made decisions by comparing the given definition with the other quadrilaterals and few students explained why they chose the definition directly. For example, one of the students performed reasoning by comparing the definition Par^5 with other quadrilaterals and concluded that the definition belongs to a parallelogram in the following way:

S_{12} : They bisect each other in a rectangle, but they are not perpendicular bisectors.

R : What do you think “bisecting” means?

S_{12} : For example, if this is 10 (showing the diagonal) then it will be bisected as 5 and 5. They are perpendicular bisectors and equal in a square. They are perpendicular bisectors in a rhombus, too, but not equal. They are not equal in a parallelogram because one is short and the other is long and they are perpendicular bisectors. They are congruent diagonals in a rectangle. Hmmm. It applies to none of them. It could be a parallelogram then.

R : For a parallelogram, you said the diagonals were perpendicular bisectors of each other.

S_{12} : Well, I changed my mind when I thought about that. They are not perpendicular bisectors. It must be a parallelogram.

Some of the students expressed their reasons for choosing the definition directly and said that the definition was sufficient for a parallelogram. For example, as shown by the following dialogue, a student made a decision based on the most critical property of a parallelogram rather than making a comparison:

S_3 : A quadrilateral with two pairs of parallel sides. A parallelogram.

R : Why do you think so?

S_3 : Because a parallelogram needs to have two pairs of parallel sides so that it consists of parallel sides.

It was determined that prototype quadrilateral images were effective in thinking process of the students who identified definitions incorrectly in this process. For the definition Par^4 based on symmetry for which all of the students performed inaccurate reasoning, the students were not able make sense of rotational symmetry and they tried to explain their ideas on drawing. One of the students (S_{13}) interpreted the definition as a square and explained this reasoning by saying, “When we rotate a square, it becomes a rhombus, and when we rotate it again, it becomes itself again”.

Identification and Reasoning Process of the Definitions of a Rectangle

Table 4 shows the students’ identification of the definitions of a rectangle among the definitions given.

Table 4. Students’ identification of rectangle definitions

		Accurate identification			**Inaccurate identification**			**Lack of comprehension of a definition**
					Inaccurate Identification	*Reasons for Inaccurate Identification*		
		Accurate identification	*Accurate identification with guidance*	*Considering the quadrilateral covered*		*Failure to recognize based on the inclusive definition*	*Considering only the quadrilateral covered*	
Rec^1	*f (%)*	8 (50)	4 (25)	10 (62,5)	4 (25)	2 (12,5)	1 (6,25)	0
Rec^2		5 (31,25)	3 (18,75)	8 (50)	7 (43,75)	5 (31,25)	2 (12,5)	1 (6,25)
Rec^3		5 (31,25)	2 (12,5)	6 (37,5)	7 (43,75)	3 (18,75)	4 (25)	2 (12,5)
Rec^4		5 (31,25)	0	3 (18,75)	10 (62,5)	3 (18,75)	7 (43,75)	1 (6,25)
Rec^5		14 (87,5)	1 (6,25)		1 (6,25)	0	0	0

As can be seen in Table 4, among the definitions given, nearly all of the students correctly identified the exclusive definition “A quadrilateral with one angle of 90° and opposite sides parallel but adjacent sides not equal in length (Rec^5)” as a rectangle. Only one student identified this definition incorrectly and decided that the definition described a parallelogram. As can be seen in the following dialogue, this student seems not to have understood the phrase “with one angle of 90°” in the definition:

S_{14} : It must be a parallelogram because the definition says it has an angle of 90°.

R : Why not the other quadrilaterals?

S_{14} : A square has all the angles of 90°, too. They are not parallel in a rhombus. It cannot be a rectangle because the definition says it has an angle of 90°.

Another definition identified accurately by nearly 75% of the students was the inclusive definition "A quadrilateral with opposite sides parallel and at least one angle of 90° (Rec^1)". In this definition, four students were observed not to have understood the phrase "with one angle of 90°" in the definition, but, with guidance, they stated that the definition described a rectangle. It was determined that the vast majority of these students recognized that this definition was an inclusive definition and that the definition also included a square.

Nearly half of the students identified the inclusive definitions "A quadrilateral with three right angles (Rec^2)" and "A quadrilateral with diagonals that are congruent and bisect each other (Rec^3)" as a rectangle. The majority of the seven students who identified the definition Rec^3 incorrectly stated that the definition only described a square. The majority of the seven students who identified the definition Rec^2 incorrectly, on the other hand, realized that the forth angle of a quadrilateral with three right angles must also be 90°, but they could not decide whether the definition described a square or a rectangle. The following dialogue from one of the interview sessions with these students could be given as an example of this situation:

S_3 : It applies to a square and a rectangle. It is valid for both of them. It does not exclusively define either of them.
R : OK. Which one of them does this definition apply to?
S_3 : It may be a square.
R : What about a rectangle?
S_3 : Well… all sides of a square are equal to each other, but all sides are not equal in a rectangle. If it were a rectangle, there would have to be another property (...) But then this applies to both a square and a rectangle, not only to one of them. But it is a definition that includes both. There must be something else so that I can distinguish them from each other. (...). So this is not a sufficient definition. It includes both. I cannot distinguish between them.

The vast majority of the students could not correctly interpret the inclusive definition "A quadrilateral with sides symmetrical to the perpendicular bisector (Rec^4)", one of them did not understand the definition at all (S_{16}), and some of them had difficulty in understanding the expressions "perpendicular bisector" and "symmetrical to the perpendicular bisector". These students often claimed that the definition referred to a square rather than a rectangle. In other words, the students took into account only a rectangle covered by the definition. The following dialogue from one of the interview sessions could be taken as an example of this situation:

S_6 : A quadrilateral with sides symmetrical to the perpendicular bisector. (…). I think it is a square.
A : Why do you think so?
S_6 : Because the perpendicular bisectors of a square are all equal.

Figure 3 shows the processes followed by the students while reasoning about the verbal definitions given for a rectangle. The students shown in red are those who incorrectly identified the given definition.

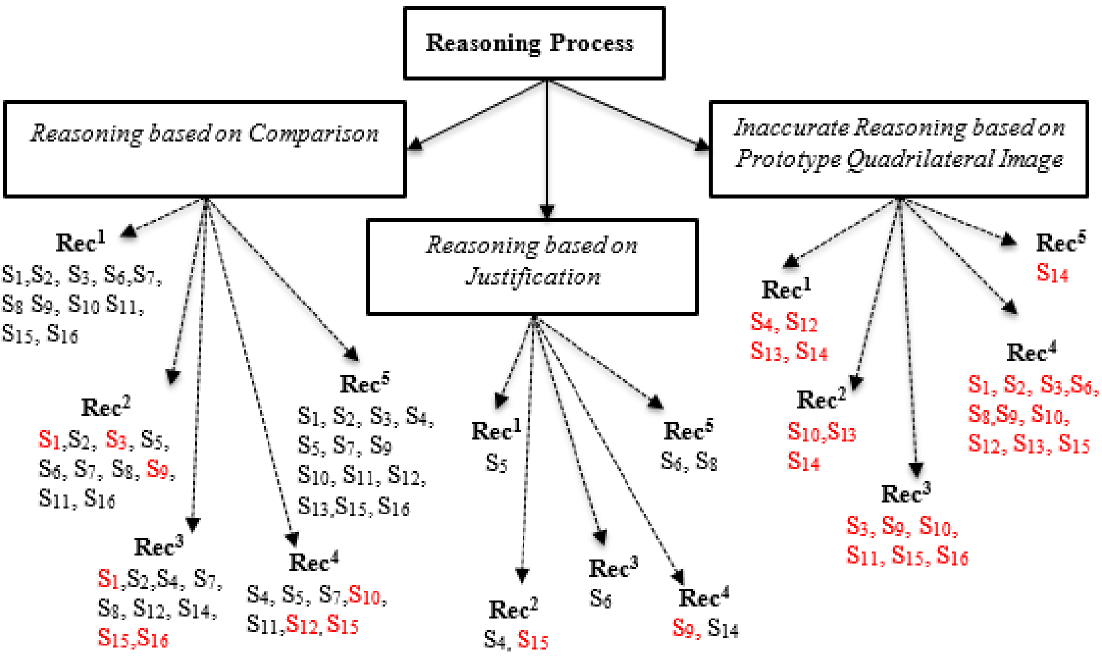


Figure 3. The students' processes of reasoning in definitions given for a rectangle

As can be seen in Figure 3, the students predominantly made their decisions by comparing the given definition with the other quadrilaterals. There were also few students who expressed the reason for choosing the definition directly and explained why the definition was sufficient for a rectangle without making any comparisons. For example, instead of making a comparison in interpreting the definition given in this process, one of the students made a decision based on the fact that a rectangle has opposite sides congruent to each other and each angle of a rectangle is 90°:

S_4 : A quadrilateral with three right angles. Then the other angle must be 90° because it is a quadrilateral. If so, this is a rectangle.

R : Why not a square?

S_4 : Because the side lengths may be different, and we cannot draw sides in different lengths because it says 90°. It cannot be a square because the sides of a square are all equal in length.

The inaccurate reasoning based on the prototype image of quadrilaterals was observed predominantly in definitions based on diagonals and symmetry. In the definition Rec^3 based on diagonals, the students drew quadrilaterals to compare diagonal lengths and made their decisions according to whether they seemed to be equal or not without emphasizing any particular property. In the definition Rec^4 based on symmetry, on the other hand, the students were confused about the concepts of congruence and symmetry/mirror image, and they searched for congruent parts for symmetry. The following dialogue from the interview session with one of the students could be taken as an example of this situation:

S_{10} : A quadrilateral with sides symmetrical to the perpendicular bisector. This could be a rectangle or a square. It may be a parallelogram as well, but not a rhombus.

R : Why do you think so?

S_{10} : Well, it does not bisect it into equal segments exactly. Hmm… I have just realized… That is also possible. I was wrong before. I misinterpreted it when I drew all of them. Then all of them would be symmetrical to the perpendicular bisectors. And the two resulting segments would be equal in length when bisected.

Identification and Reasoning Process of the Definitions of a Rhombus

Table 5 shows the students' identification of the definitions of a rhombus among the definitions given to verbally.

Table 5. Students' identification of rhombus definitions

		Accurate identification			**Inaccurate identification**			**Lack of comprehension of a definition**
					Inaccurate Identification	*Reasons for Inaccurate Identification*		
		Accurate identification	*Accurate identification with guidance*	*Considering the quadrilateral covered*		*Failure to recognize based on the inclusive definition*	*Considering only the quadrilateral covered*	
Rh^1	*f (%)*	6 (37,5)	4 (25)	10 (62,5)	6 (37,5)	1 (6,25)	5 (31,25)	0
Rh^2		9 (56,25)	1 (6,25)	9 (56,25)	6 (37,5)	1 (6,25)	5 (31,25)	0
Rh^3		7 (43,75)	0	4 (25)	8 (50)	3 (18,75)	5 (31,25)	1 (6,25)
Rh^4		4 (25)	1 (6,25)	3 (18,75)	10 (62,5)	5 (31,25)	4 (25)	1 (6,25)
Rh^5		16 (100)	0		0	0	0	0

As can be seen in Table 5, all of the students correctly identified the exclusive definition "A quadrilateral with four equal sides and no right angles (Rh^5)" as a rhombus. Among the definitions given, more than half of the students correctly identified the inclusive definition "A quadrilateral with four equal sides (Rh^1)", which is based on the side property of a rhombus. In addition, all of the students with correct answers stated that the definition also applied to a square. Among the six students with incorrect answers, on the other hand, five students matched the definition with only a square. The following dialogue from one of the interview sessions could be taken as an example of this situation:

S_4 : A quadrilateral with four equal sides. This must be a square.

R : OK. Are there any other quadrilaterals?

S_4 : Rhombus!
R : Then why did you choose a square over a rhombus?
S_4 : Well, a square is what you think of first when it is about all sides equal in length.

One of the students who incorrectly identified this definition failed to determine whether it applied to a square or to a rhombus:

S_3 : A quadrilateral with four equal sides. This is valid for a square and a rhombus, but not for a rectangle and a parallelogram. In this case, this definition should be valid for a square and a rhombus. Because all the side lengths are equal in both of them. They (a square and a rhombus) change depending on their angles. And because the definition does not state anything like depending on angles, it applies to both.
A : Are all quadrilaterals with four equal sides squares?
S_3 : No. This is also the type with a rhombus. I am totally confused now. It applies to both. It should say something else so that I can distinguish them from each other.

The definition "A quadrilateral with opposite angles equal and all sides equal length (Rh^2)", which is based on angle property, was another definition that more than half of the students (including those guided by the researcher) identified correctly. Among the students who incorrectly identified this definition, five students stated that it described only a square. These students were observed to associate the congruence of corresponding angles with all the angles of a square being 90°.

More than half of the students were unable to accurately interpret the inclusive definition "A quadrilateral in which the diagonals are perpendicular bisectors of each other (Rh^3)", which is based on diagonal property, one of the students (S_6) did not understand the definition at all, and some other students had difficulty in understanding the phrase "perpendicular bisectors" in the definition. The following dialogue with one of the students with incorrect answers could be given as an example of this situation:

S_4 : A quadrilateral in which the diagonals are perpendicular bisectors of each other.
R : What does "perpendicular bisector" mean?
S_4 : (drawing). It will be 90° when we draw like this. It cannot be a rectangle, because these (angles) are different. It can be a square. It cannot be a parallelogram, either because there is no perpendicular bisector here. There is perpendicular bisector in a rhombus, too. I think this definition could apply to both.

Another definition that the vast majority of the students failed to interpret correctly and, in fact, two of the students were not able understand at all was the definition "A quadrilateral that is symmetric across both diagonals (Rh^4)", which is based on symmetry property. Most of the students who identified this definition correctly stated that it also applied to a square as well, but it was not a sufficient definition for a square. The following dialogue from one of the interview sessions could be taken as an example of this situation:

S_7 : A quadrilateral in which the diagonals are perpendicular bisectors of each other. They are perpendicular bisectors of each other in both a square and a rhombus. I would say a rhombus because this is not a sufficient definition for a square.
R : Why do you think it is not a sufficient definition for a square?
S_7 : Because in a square, they should be perpendicular bisectors of each other and these (diagonals) should be equal in length.

Some of the students who inaccurately interpreted this definition could not recognize a rhombus based on the inclusive definition Rh^4 and some others stated that this definition applied only to a square. One of the students interpreted the concept of symmetry by associating it with bisecting into equal segments. The following dialogue from the interview session with this student is presented as an example of this situation:

S_3 : A quadrilateral that is symmetric across both diagonals. That is, it will be reflected like a mirror according to the diagonals. Because here is 90 (a corner angle of a square) and there is 45, we have two congruent triangles when it is divided by the diagonal. If it goes through here like this in a rectangle (reflecting the vertex across the diagonal), it is not the type in a rectangle. It is not the case here, either (showing a parallelogram). Something else appears here like in a rectangle. It is symmetrical in a rhombus, too. Then I would say it is a square and a rhombus.
R : For which one do you think it is a sufficient definition?
S_3 : But both are symmetrical. But since all sides and all angles are equal, when we compare the symmetry across this diagonal in a square and the symmetry across that diagonal in a rhombus, they are the same. But the lengths of the diagonals of a rhombus are different, so the

symmetry across this diagonal is not the same as the symmetry across the other diagonal. Then it can only be a square.

In their processes of reasoning in interpreting definitions given for a rhombus, the students mainly followed three ways and these processes are presented in Figure 4.

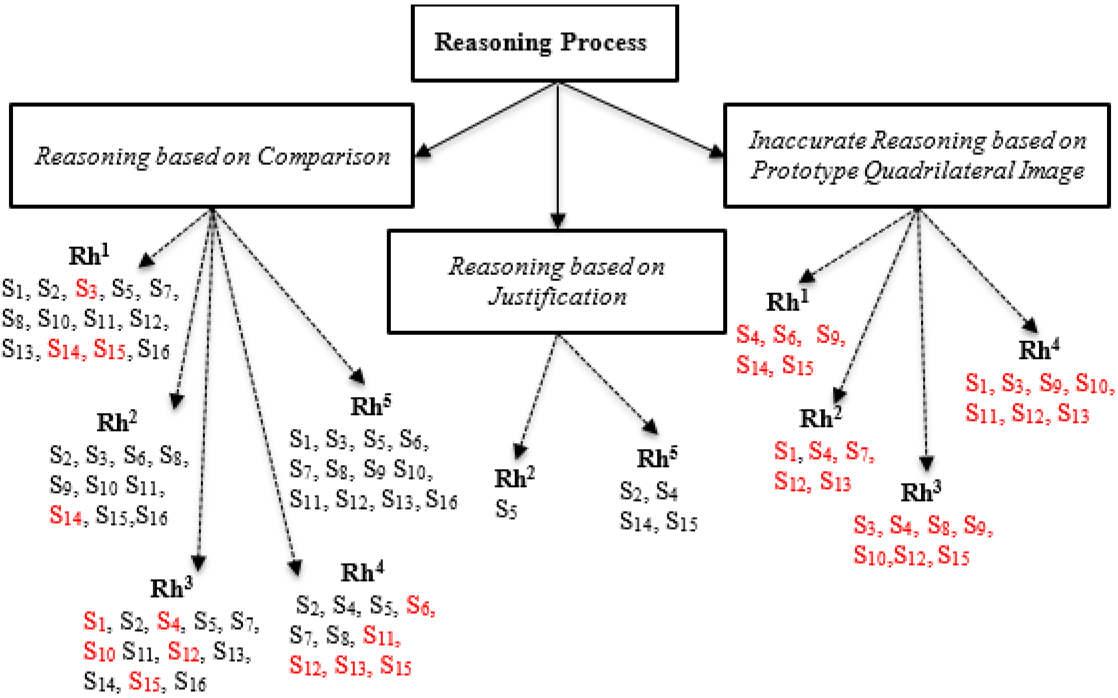
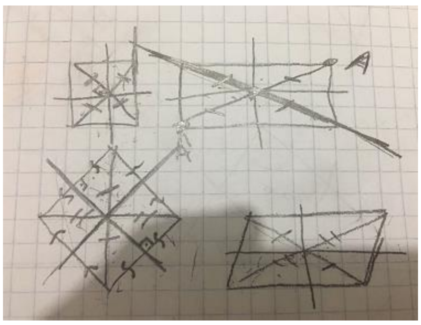


Figure 4. The students' processes of reasoning in definitions given for a rhombus

As can be seen in Figure 4, the students primarily made their decisions by comparing the given definition with the other quadrilaterals and few students explained why they chose the definition directly. For example, one of the students who stated their reasons for choosing the definition directly (S_2) explained that he made a decision based on the most critical property of a rhombus and added, "A rhombus has four equal sides and no right angles". It was determined that prototype quadrilateral images were effective in the thinking processes of the students with incorrect answers. In line with this tendency, these students interpreted a rhombus required in almost each inclusive definition as a square and drew a square. For example, focused entirely on the drawing of the shape for the definition Rh^4, one of the students (S_{11}) said, "*I think of a square and a rectangle can have perpendicular bisectors because their angles are 90°*" whereas another student (S_3) thought of a square together with a rhombus. The drawing of and dialogue with this student is presented as an example:



S_3 : A quadrilateral in which the diagonals are perpendicular bisectors of each other. A square!
R : Why do you think so?
S_3 : (drawing) The diagonals are right. But this is not the type of rectangle.
R : Is there another quadrilateral that meets this condition?
S_3 : Not a parallelogram, (asking for a ruler and drawing) but it is a perpendicular bisector in a rhombus. So I would say it is a rhombus or a square.
R : OK. Which one does the definition match with?
S_3 : I think it is a square.

R : Why do you think so?
S_3 : Well, a rhombus and a square are almost the same things.
R : In what way?
S_3 : Now the sides are equal in this one and so are they in that one. But they are in different orientations .

Identification and Reasoning Process of the Definitions of a Square

Table 6 shows the students' identification of the definitions of a square among the definitions given to verbally.

Table 6. Students' identification of square definitions

		Accurate identification		Inaccurate identification		Lack of comprehension of a definition
		Accurate identification	*Accurate identification with guidance*	*Inaccurate Identification*	*Reasons for Inaccurate Identification*	
Squ^1	*f (%)*	12 (75)	1 (6,25)	2 (12,5)	2 (12,5)	1 (6,25)
Squ^2		13 (81,25)	0	1 (6,25)	1 (6,25)	2 (12,5)
Squ^3		14 (87,5)	1 (6,25)	1 (6,25)	1 (6,25)	0
Squ^4		9 (56,25)	2 (12,5)	4 (25)	3 (18,75)	1 (6,25)

As can be seen in Table 6, all of the students correctly interpreted the definition "A quadrilateral with diagonals that are congruent and perpendicular bisectors of each other (Squ^3)", which based on diagonal property. One of the students (S_{16}), who misinterpreted the definition Squ^3, could not recognize a square based on the definition. On the other hand, the vast majority of the students (about 81% of them) accurately identified the definition "A quadrilateral with equal sides and one right angle (Squ^1)", which is based on side property, and the definition "A regular quadrilateral (Squ^2)", which is based on angle property of a square. It was also observed that two students couldn't understand the definition Squ^2 and only one student (S_{11}), who could not recognize a square based on the definition, misinterpreted this definition. This student took into account only the side length for the expression "a regular quadrilateral" in the definition and said, "It must be describing a square and a rhombus. Because all the sides are equal".

One of the students could not understand the definition Squ^1, two students misinterpreted it and they were not able to decide whether the definition described a square or a rhombus. One interesting result about this definition was that four students could not understand the expression "one right angle" in the definition and they thought only one of the angles was 90°. It was observed that only one of the students (S_6) could not understand the definition "A quadrilateral that is symmetrical to the diagonals and perpendicular bisectors (Squ^4)", which is based on symmetry property, but nearly 75% of the other students stated that the definition described a square. However, the vast majority of the students with incorrect answers failed to recognize a square based on the definition Squ^4.

The students' processes of reasoning in interpreting the verbal definitions given for a square are presented in Figure 5. As can be seen in Figure 5, like the case with the other quadrilaterals, the students primarily made decisions by comparing the given definition with the other quadrilaterals. Few students explained why they chose the definition directly especially for those based on the side and angle properties of a square and why the definition was sufficient for a square. The following dialogue with one of these students could be given as an example of this situation:

S_2 : A regular quadrilateral. I think this is a square.
R : Why do you think so?
S_2 : Because in a square all sides are equal to each other and the interior angles are 90°. That is why I say it is a square.

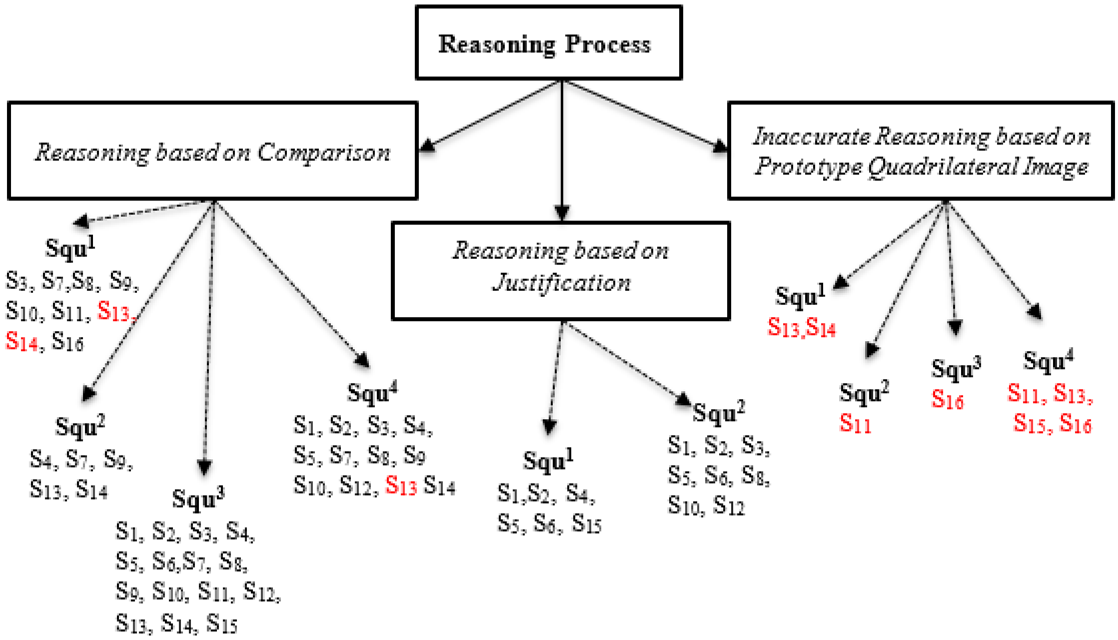


Figure 5. The students' processes of reasoning in definitions given for a square

It was determined that prototype quadrilateral images were effective in thinking process of students with incorrect answers. For example, three students (S_{11}, S_{15}, S_{16}) thought that a parallelogram, a rhombus and a square are symmetrical across their diagonals and perpendicular bisectors. In the same way, one student (S_{13}) thought that a rectangle, a rhombus and a square are symmetrical across their diagonals and perpendicular bisectors. The following dialogue from the interview sessions with one of the students, who interpreted the concept of symmetry by associating it with bisecting into equal parts, is presented as an example of this situation:

S_{13} : A quadrilateral that is symmetrical to the diagonals and perpendicular bisectors. It must be a square because the diagonals and perpendicular bisectors are divided into two equally.

R : What does symmetrical mean?

S_{13} : Bisecting a part into two equal parts. Dividing into two equally. It could be a rectangle, too because its diagonals and perpendicular bisectors are bisected equally. But it cannot be a parallelogram.

R : Why do you think so?

S_{13} : The sides are inclined. It can also be a rhombus because the diagonals and perpendicular bisectors are symmetrical. They are bisected. So I would say it could be a rhombus, rectangle and square.

An Overview of Interpretation of Parallelograms Family Definitions

The verbal definitions of the parallelogram family given to the students were presented in 5 different alternative ways as inclusive or exclusive definitions based on side, angle, diagonal and symmetry properties. In general, the definitions were interpreted most accurately by the students were exclusive definitions as presented in Table 7.

Table 7. Accurate identification of parallelograms definitions by the students based on properties

	Parallelogram	Rectangle	Rhombus	Square
Side	93,75%	75%	62,5%	81,25%
Angle	87,5%	50%	62,5%	81,25%
Diagonal	12,5%	43,75%	43,75%	93,75%
Symmetry	0%	31,25%	31, 25%	68,75%
Exclusive	93,75%	93,75%	100%	

In general, the parallelograms definitions were interpreted most accurately by the students were exclusive definitions. As can be seen in Table 7, comparison of inclusive definitions based on side, angle, diagonal and symmetry properties showed that the definitions understood most by the students were the ones based on side and angle properties of the quadrilaterals and the most challenging definitions were based on the diagonal and symmetry properties. This situation only changes for a square, and, unlike the other quadrilateral definitions, the definition of a square based on diagonal properties was interpreted more accurately by the students.

Results, Discussion and Recommendations

Making a definition and interpreting a given definition and classifying based on definitions are intertwined mathematical skills that are related to each other and require higher-order thinking. Especially for students who do not learn geometry in a deductive way at middle school level (Heinze & Ossietzky, 2002), interpreting quadrilateral definitions and identifying related quadrilaterals based on alternative definitions is an important cognitive challenge that develops geometric thinking. This study investigated how students interpreted the verbal definitions given for the parallelogram family without a visual support and their reasoning in this interpretation. The most important result of this study was that the students more accurately interpreted the exclusive definition and inclusive definition based on side properties given for a parallelogram, the exclusive definitions for a rectangle and a rhombus, and the definition based on diagonal properties for a square. The students' highly accurate interpretation of the exclusive definitions for a rectangle and a rhombus in particular is compatible with the results from other studies about quadrilateral classification (de Villiers, 1994, Duatepe-Paksu, 2016; Ulusoy & Çakıroğlu, 2017), which showed that students preferred partition classification. In fact, the students' tendency towards exclusive definitions is only natural when we consider the results suggesting that middle school mathematics teacher candidates also preferred to use partition classification (Türnüklü, Gündoğdu-Alaylı & Akkaş, 2013), it is not surprising that students move to exclusive definitions.

Another important result was that the vast majority of the students who accurately interpreted the inclusive definitions for a parallelogram based on side and angle properties suggest that these definitions also apply to a rectangle, a square and a rhombus. This result shows that the students knew the quadrilaterals covered by a parallelogram and interpreted the inclusive definitions "*a quadrilateral with two pairs of parallel sides*" and "*a quadrilateral with opposite angles equal*" to be sufficient. This result is confirmed by the findings suggesting that the definition of a parallelogram as "*a quadrilateral with two pairs of parallel sides*" is often made by students (Fujita & Jones, 2007) and even by mathematics teacher candidates (Ozdemir Erdoğan & Dur, 2014). In addition, presentation of verbal definitions given for a parallelogram based on side and angle properties but without a visual support, like the case in a study by Haj-Yahya and Hershkowitz (2013), could have let the students identify the inclusive relationships more accurately. As emphasized by Heinze and Ossietzky (2002), among parallelogram definitions, those based on diagonal and symmetry properties cannot accurately be interpreted without going through an analytical thinking process. Interpreting the definition "*a quadrilateral that has rotational symmetry*" as a parallelogram requires making an inference that a parallelogram has opposite sides that are parallel and equal with each other, its corresponding angles are congruent and its diagonals bisect each other. Unfortunately, none of the students were able to interpret the definition based on symmetry property as a parallelogram, in fact three of them did not understand the definition at all, and most of them interpreted this definition to describe a square, a rhombus and a rectangle. Moreover, the vast majority of the students failed to interpret the definition "a quadrilateral whose diagonals bisect each other" as a parallelogram. Also, the vast majority of the students with wrong answers could not realize that the necessary and sufficient condition in this definition described a parallelogram and recognize a parallelogram based on the definition.

For definitions of a rectangle and a rhombus, the students primarily interpreted the definitions based on side and angle properties accurately but few students were able to interpret the definitions based on diagonal and symmetry properties. The vast majority of the students who misinterpreted the definition "a quadrilateral with sides symmetrical to the perpendicular bisector" in particular identified the definition as a square and they took into account only the quadrilateral it covered. This result also suggests that the students establish an inverse hierarchical relationship between a rectangle and a square. Similar results were obtained in both studies conducted with students (de Villiers, 1994; Heinze & Ossietzky, 2002) and studies conducted with middle school mathematics teacher candidates (Toluk & Olkun, 2004, Türnüklü, 2014). Some of the students were not able to deduce what the necessary and sufficient condition of a rectangle is based on the given rectangle definition "a quadrilateral with three right angles" and they could not decide whether the definition described a rectangle or a square. Similarly, some of the students failed to identify the rhombus definition "a quadrilateral in which the diagonals are perpendicular bisectors of each other". In addition, the fact that some of the students identified the given definition as a square for all the inclusive rhombus definitions based on side, angle, diagonal and symmetry properties except for the exclusive definition suggests that these students were misguided by the inverse hierarchical relationship between a rhombus and a square. These results also suggest that some of the students, who established an inverse hierarchical relationship, did not see a square as a special type of a rhombus and a special type of a rectangle. This result is in line with other study results (Fujita & Jones, 2007; Aktas & Aktas, 2012, Türnüklü & Berkün, 2013), which revealed that students had difficulty in realizing that a square is a special type of a rhombus and rectangle. Also, unlike the case with other quadrilaterals, the square definition "a quadrilateral with diagonals that are congruent and perpendicular bisectors of each other", which is based on diagonal properties, was interpreted more accurately by the students. This result could be attributed to the fact that a square is a quadrilateral that students most frequently encounter beginning from elementary school.

While interpreting the definitions, some of the students failed to make sense of the critical expressions embedded in the inclusive definitions given. In the definitions based on angle, diagonal and symmetry properties given for almost all of the quadrilaterals, some of the students had difficulty in understanding the expressions such as rotational symmetry, bisecting, bisecting perpendicularly, having at least one angle of 90°, having one right angle, perpendicular bisector and symmetrical across the perpendicular bisector. This difficulty in understanding the language of mathematics may also have affected the interpretation of the definitions in this study. In fact, Heinze and Ossietzky (2002) argued that this challenge influences determining the necessary and sufficient conditions in definitions. The difficulty faced by the students in our study in interpreting the inclusive definitions particularly based on diagonal and symmetry properties could have been caused by their failure to make sense of the properties of the critical expressions in the definitions such as "perpendicular bisector", "bisecting" and "rotational symmetry". The students especially had difficulty in understanding and identifying the expressions "having at least one angle of 90°" and "having one right angle", and they associated the case in which a quadrilateral has one right angle with the case in which it has only one angle of 90°. Expressions that are already ambiguous in students' minds also prevent a deep geometric understanding. This situation is supported by the fact that students could not use appropriate mathematics language and could not make appropriate definitions when they were asked to define a parallelogram (Aktaş, 2016) or another object (Bozkurt & Koç, 2012). It is therefore important that teachers take care of the mathematical language used by both themselves and their students in teaching geometric concepts. It may be advisable to include informative and meaningful activities for all the terms used to enhance the students' understanding in the teaching-learning process.

The students performed three types of reasoning while interpreting the definitions given "reasoning based on comparison," "reasoning based on justification" and "inaccurate reasoning based on prototype quadrilateral image". The students were not expected to perform any drawings for the definitions given, but all of them generally made prototype drawings of the relevant quadrilateral while interpreting the definitions. There are similar research results about students making drawings although they were not expected to do so (Aktaş & Aktaş, 2012). Students making drawings in order to reveal the relationships in the definition is only natural, but the students' drawings in our study were prototypes shapes and this could have caused them to perform inaccurate reasoning. In fact, students' drawing a rhombus as a square caused them either to interpret the given rhombus definition as a square or to fail to distinguish a rhombus from a square by making them confused about the two shapes. As stated by Ulusoy and Çakıroğlu (2017), this situation may be related to the textbooks given to students as well as the examples teachers use in the course of a lesson. There are also study results which showed that prototypes are prevalent in both teachers' and students' sample spaces of quadrilaterals (Okazaki & Fujita, 2007; Ozdemir Erdoğan & Dur, 2014; Türnüklü, Gündoğdu-Alaylı & Akkaş, 2013). In our study, the definitions influenced by the students' prototype quadrilateral images were predominantly based on diagonal and symmetric properties. In the case of definitions based on diagonal properties, the students performed reasoning through prototype drawings without considering any angle, side or diagonal properties, whereas they focused on congruent parts appearing in the relevant quadrilateral in definitions based on symmetry properties. This result about students with inaccurate interpretation of definitions is not surprising at all when we consider the fact that there are students, and even teacher candidates, who state that the concept of symmetry is associated with congruent parts and that a parallelogram and a rectangle are symmetrical across the diagonals (Köse & Özdaş, 2009; Karadeniz et al., 2015). In fact, the concept of symmetry deepens students' understanding of mathematical concepts as well as their problem solving skills (Leikin, Berman & Zaslavsky, 2000). Students' correct interpretation of definitions based on symmetry and diagonal properties will support their ability to express the properties of the quadrilateral in question. For this reason, it is recommended that teachers should not depend on a single definition in the classroom, but also include alternative definitions.

This study focused on introducing definitions as inclusive and exclusive definitions, and it did not include any definitions based on hierarchy and family structure such as "a rectangle is a parallelogram whose diagonals are congruent". On the other hand, future research can investigate how students interpret definitions that are hierarchical and based on family structure and their reasoning in this interpretation process.

References

Aktaş, M. C. (2016) Turkish high school students' definitions for parallelograms: Appropriate or inappropriate? *International Journal of Mathematical Education in Science and Technology*, *47*(4), 583-596.

Aktaş, D. Y., & Aktaş, M. C. (2012). Eighth grade students' understanding hierarchical classification of quadrilaterals. *Elementary Education Online, 11*(3), 714-728.

Bozkurt, A. & Koç, Y. (2012). Investigating first year elementary mathematics teacher education students' knowledge of prism. *Educational Sciences: Theory & Practice*, *12* (4), 2941-2952.

Casa, T. M., & Gavin, M. K. (2009). Advancing elementary school students' understanding of quadrilaterals. In T. V., Craine, & R. Rubenstein (Eds.), *Understanding Geometry for a Changing World 71st Yearbook* (pp. 205-219). Reston, VA: National Council of Teachers of Mathematics.

Çakıroğlu, E. (2013). Matematik kavramlarının tanımlanması. In İ. Zembat, M.F. Özmantar, E. Bingölbali, H. Şandır & A. Delice (Eds.), *Tanımları ve tarihsel gelişimleriyle matematiksel kavramlar* (pp. 2-12). Ankara: Pegem Akademi.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds), *Handbook of research design in mathematics and science education* (pp. 547–589). London: Lawrence Erlbaum Associates, Publishers.

de Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals. *Learning of Mathematics*, *14*(1), 11-18.

de Villiers, M. (1998). To teach definitions in geometry or teach to define? In A. Olivier & K. Newstead (Eds.), *Proceedings of the Twenty-second International Conference for the Psychology of Mathematics Education* (vol 2, pp. 248-255). Stellenbosch: University of Stellenbosch.

de Villiers, M., Govender, R., & Patterson, N. (2009). Defining in geometry. In T. V. Craine & R. Rubenstein (Eds.), *Understanding Geometry for a Changing World 71st Yearbook* (pp. 189-203). Reston, VA: National Council of Teachers of Mathematics.

Duatepe-Paksu, A. (2016). Examining quadrilaterals by paper folding. *Journal of Inquiry Based Activities*, *6*(2), 80-88.

Fujita, T. (2008). Learners' understanding of the hierarchical classification of quadrilaterals. In Joubert, M. (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *28*(2), 31-36.

Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *The Journal of Mathematical Behavior*, *31*(1), 60-72.

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education*, *9*(1-2), 3-20.

Ginsburg, H.P. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For The Learning of Mathematics, 1*(3), 4-11.

Güven, B. & Okumuş, S. (2011). 8th grade Turkish students' van Hiele levels and classification of quadrilaterals. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (vol 2, pp. 473-480). Ankara, Turkey: PME.

Haj-Yahya, A., & Hershkowitz, R. (2013). When visual and verbal representations meet the case of geometrical figures. In Lindmeier, A. M. & Heinze, A. (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (vol 2, pp. 409-416). Kiel, Germany: PME.

Heinze, A., & Ossietzky, C. (2002). "…Because a square is not a rectangle" students' knowledge of simple geometrical concepts when starting to learn proof. In A. Cockburn & E. Nardi (Eds.), *Proceedings of The 26th Conference of the International Group for the Psychology of Mathematics Education*, 3, 81-88.

Jamison, R. E. (2000). Learning the language of mathematics. *Language and Learning Across the Disciplines, 4*(1), 45-54.

Jones, K., & Tzekaki, M. (2016). Research on the teaching and learning of geometry. In A. Gutiérrez, G. Leder & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education: The Journey Continues* (pp. 109-149). Rotterdam: Sense.

Karadeniz, M. H., Baran, T., Bozkuş, F., & Gündüz, N. (2015). Difficulties of prospective elementary mathematics teachers' regarding to reflection symmetry. *Turkish Journal of Computer and Mathematics Education*, *6*(1), 117-138.

Kondratieva, M. F., & Radu, O. G. (2009). Fostering connections between the verbal, algebraic, and geometric representations of basic planar curves for student's success in the study of mathematics. *The Mathematics Enthusiast*, *6*(1-2), 213-238.

Köse, N. Y. & Özdaş, A. (2009). How do the fifth grade primary school students determine the line of symmetry in various geometrical shapes using Cabri Geometry software? *Elementary Education Online, 8*(1), 159-175.

Leikin, R., Berman, A. & Zaslavsky, O. (2000). Learning through teaching: The case of symmetry. *Mathematics Education Research Journal*, *12* (1), 18-36.
Merriam, S. B. (2009). *Qualitative research and case study applications in education*. San Francisso: Jossey-Bass.
Milli Eğitim Bakanlığı Talim ve Terbiye Kurulu Başkanlığı (MoNE). (2013). *Ortaokul Matematik Dersi Öğretim Programı (Secondary School Mathematics Curriculum)*. Ankara: MEB.
Miles M., & Huberman, M. (1994). *An expanded sourcebook qualitative data analysis* (2nd Ed.). California: Sage Publications.
Monroe, E. E., & Panchyshyn, R. (1995). Vocabulary considerations for teaching mathematics. *Childhood Education*, *72*(2), 80-83.
Okazaki, M. & Fujita, T. (2007). Prototype phenomena and common cognitive paths in the understanding of the inclusion relations between quadrilaterals in Japan and Scotland. In J. H. Woo, H. C. Lew, K. Park, D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol 4, pp.41-48), Seoul: PME.
Ozdemir Erdogan, E., & Dur, Z. (2014). Preservice mathematics teachers' personal figural concepts and classifications about quadrilaterals. *Australian Journal of Teacher Education*, 39(6), 107-133.
Öztoprakçı, S. & Çakıroğlu, E. (2013). Dörtgenler (Quadrilaterals). In İ. Zembat, M.F. Özmantar, E. Bingölbali, H. Şandır & A. Delice (Eds.), *Tanımları ve tarihsel gelişimleriyle matematiksel kavramlar* (pp. 249-272), Ankara: Pegem Akademi.
Rubenstein, R. N., & Thompson, D. R. (2002). Understanding and support children's mathematical vocabulary development. *Teaching Children Mathematics*, *9*(2), 107-112.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151-169.
Toluk, Z., & Olkun, S. (2004). Pre-Service elementary teachers' geometric thinking levels. *Education and Science, 29* (134), 55-60.
Türnüklü, E., & Berkün, M. (2013). 5th and 7th grade primary students' strategies of polygons classification. *Kastamonu Education Journal*, *21*(1), 337-356.
Türnüklü, E., Gündoğdu Alaylı, F. & Akkaş, E. N. (2013). Investigation of prospective primary mathematics teachers' perceptions and images for quadrilaterals. *Educational Sciences: Theory & Practice*, *13* (2), 1213-1232.
Türnüklü, E. (2014). Construction of inclusion relations of quadrilaterals: Analysis of pre-service elementary mathematics teachers' lesson plans. *Education and Science*, *39* (173), 197-207.
Ulusoy, F. (2015). A meta-classification for students' selections of quadrilaterals: The case of trapezoid. In K. Krainer & N. Vondrová (Eds.), *Cerme 9-Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 598-604). Prague: Czech Republic.
Ulusoy, F., & Çakıroğlu, E. (2017). Middle school students' types of identification for parallelogram: Underspecification and overgeneralization. *Abant İzzet Baysal University Journal of Faculty of Education,17* (1), 457- 475.
Usiskin, Z., Griffin, J., Witonsky, D., & Willmore, E. (2008). *The Classification of Quadrilaterals: A Study of Definition*. Charlotte, NC: Information Age Publishing.
Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, *36* (4), 317-346.

Author Information

Nilüfer Yavuzsoy Köse
Anadolu University
Faculty of Education, Department of Mathematics and Science Education, Eskisehir
Turkey
Contact e-mail: *nyavuzsoy@anadolu.edu.tr*

Tuğba Y. Yılmaz
Ministry of National Education
Eskisehir
Turkey

Demet Yeşil
Ministry of National Education
Eskisehir
Turkey

Duygu Yıldırım
Ministry of National Education
Eskisehir
Turkey